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Constructing Quantum Error-Correcting Codes for $p^n$-State Systems from Classical Error-Correcting Codes*

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SUMMARY We generalize the construction of quantum error-correcting codes from $F_4$-linear codes by Calderbank et al. to $p^n$-state systems. Then we show how to determine the error from a syndrome. Finally we discuss a systematic construction of quantum codes with efficient decoding algorithms.

key words: quantum codes, stabilizer codes, additive codes, self-orthogonal codes

1. Introduction

Quantum error-correcting codes have attracted much attention. Among many research articles, the most general and systematic construction is the so called stabilizer code construction [6] or additive code construction [2], which constructs a quantum error-correcting code as an eigenspace of an Abelian subgroup $S$ of the error group. Thereafter Calderbank et al. [3] proposed a construction of $S$ from an additive code over the finite field $F_4$ with 4 elements.

These constructions work for tensor products of 2-state quantum systems. However Knill [8], [9] and Rains [13] observed that the construction [2], [6] can be generalized to $n$-state systems by an appropriate choice of the error basis. Rains [13] also generalized the construction [3] using additive codes over $F_4$ to $p$-state quantum systems, but his generalization does not relate the problem of quantum code construction to classical error-correcting codes. We propose a construction of quantum error-correcting codes for $p^n$-state systems from classical error-correcting codes which is a generalization of [3].

Throughout this paper, $p$ denotes a prime number and $m$ a positive integer. This paper is organized as follows. In Sect. 2, we review the construction of quantum codes for nonbinary systems. In Sect. 3, we propose a construction of quantum codes for $p$-state systems from classical error-correcting codes over $F_4$. In Sect. 4, we propose a construction of quantum codes for $p^n$-state systems from classical linear codes over $F_{p^m}$. In Sect. 5, we discuss a systematic construction of quantum codes with efficient decoding algorithms.

2. Stabilizer Coding for $p^n$-State Systems

2.1 Code Construction

We review the generalization [8], [9], [13] of the construction [2], [6]. First we consider $p$-state systems. We shall construct a quantum code $Q$ encoding quantum information in $p^k$-dimensional linear space into $C^n$.

$Q$ is said to have minimum distance $d$ and said to be an $[[n, k, d]]_p$ quantum code if it can detect up to $d - 1$ quantum local errors. Let $\lambda$ be a primitive $p$-th root of unity, $C_p$, $D_{\lambda}$, $p \times p$ unitary matrices defined by $(C_p)_{ij} = \delta_{j-1, i \mod p}$, $(D_{\lambda})_{ij} = \lambda^{i-j} \delta_{ij}$. Notice that $C_2$ and $D_{-1}$ are the Pauli spin matrices $\sigma_x$ and $\sigma_z$. We consider the error group $E$ consisting of $\lambda^{i}w_1 \cdots \otimes w_n$, where $j$ is an integer, $w_i = C_{\lambda}^a D_{\lambda}^b$ with some integers $a, b$.

For row vectors $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, $(a,b)$ denotes the concatenated vector $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ as used in [3]. For vectors $(a|b), (a'|b') \in F_{p^n}$, we define the alternating inner product

$$((a|b), (a'|b')) = (a,b) - \langle a', b \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $F_p^n$. For $a = (a_1, \ldots, a_n) \in F_{p^n}$, we define

$$X(a) = C_{\lambda}^{a_1} \otimes \cdots \otimes C_{\lambda}^{a_n},$$

$$Z(a) = D_{\lambda}^{a_1} \otimes \cdots \otimes D_{\lambda}^{a_n}.$$

Then we have

$$X(a) Z(b) X(a') Z(b') = \lambda^{\langle a,a' \rangle - \langle a,b \rangle} X(a') Z(b') X(a) Z(b).$$

For $(a|b) = (a_1, \ldots, a_n, b_1, \ldots, b_n) \in F_{p^n}$, we define the weight of $(a|b)$ to be

$$w \{i \mid a_i \neq 0 \text{ or } b_i \neq 0 \}.$$

Theorem 1: Let $C$ be an $(n-k)$-dimensional $F_{p^n}$-linear subspace of $F_{p^n}$ with the basis $(a_1|b)_1, \ldots, (a_{n-k}|b_{n-k})$, $C^\perp$ the orthogonal space of $C$ with respect to the inner product (1). Suppose that $C \subseteq C^\perp$ and the minimum weight (3) of $C^\perp \setminus C$ is $d$. Then...
the subgroup $S$ of $E$ generated by $\{X(a_1)Z(b_1), \ldots, X(a_{n-k})Z(b_{n-k})\}$ is Abelian, and an eigenspace of $S$ is an $[[n, k, d]]_p$ quantum code.

Next we consider quantum codes for $p^m$-state systems, where $m$ is a positive integer. But the code construction for $p^m$-state systems is almost the same as that for $p$-state systems, because the state space of a $p^m$-state system can be regarded as the $m$-fold tensor products of that of a $p$-state system. We shall construct a quantum code encoding quantum information in $p^{mk}$-dimensional linear space into $\mathbb{C}^{p^{mn}}$. For $(a|b) = (a_{1,1}, a_{1,2}, \ldots, a_{1,m}, a_{2,1}, \ldots, a_{n,m}, b_{1,1}, \ldots, b_{n,m}) \in \mathbb{F}_p^{2mn}$, we define the weight of $(a|b)$ to be

$$w = \sum_{i=1}^{n} (\sum_{j=1}^{m} a_{i,j}b_{i,j})$$

Corollary 2: Let $C$ be an $(mn - mk)$-dimensional $\mathbb{F}_p$-linear subspace of $\mathbb{F}_p^{2mn}$ with the basis $\{a_1|b_1\}, \ldots, (a_{mn-mk}|b_{mn-mk})$. $C^\perp$ the orthogonal space of $C$ with respect to the inner product (1). Suppose that $C \subseteq C^\perp$ and the minimum weight (2) of $C^\perp \setminus C$ is $d$. Then the subgroup $S$ of $E$ generated by $\{X(a_1)Z(b_1), \ldots, X(a_{mn-mk})Z(b_{mn-mk})\}$ is Abelian, and an eigenspace of $S$ is an $[[n, k, d]]_p$ quantum code.

2.2 Error Correction Procedure

In this subsection we review the process of correcting errors. Let $H = C^\perp, H^{\otimes n} \supset Q$ the quantum code constructed by Theorem 1, and $H_{env}$ the Hilbert space representing the environment. Suppose that we send a codeword $|\psi\rangle \in Q$, that the state of the environment is initially $|\psi_{env}\rangle \in H_{env}$, and that we receive $|\psi'\rangle \in H^{\otimes n} \otimes H_{env}$. Then there exists a unitary operator $U$ such that

$$|\psi'\rangle = U(|\psi\rangle \otimes |\psi_{env}\rangle).$$

If $U$ acts nontrivially on $\tau$ ($0 \leq \tau \leq n$) subsystems among $n$ tensor product space $H^{\otimes n}$, then $\tau$ is said to be the number of errors.

We assume that $2\tau + 1 \leq d$, where $d$ is as in Theorem 1. If we measure each observable in $H^{\otimes n}$ whose eigenspaces are the same as those of $X(a_i)Z(b_i)$ for $i = 1, \ldots, n - k$, $X(a_i)Z(b_i)$ as defined in Theorem 1, then the entangled state $|\psi'\rangle$ is projected to $A_i|\psi\rangle \otimes |\psi_{env}\rangle$, for some $A \in E$ and $|\psi_{env}\rangle \in H_{env}$, by the measurements. By the measurement outcomes we can find a unitary operator $A' \in E$ such that $A'A|\psi\rangle = |\psi\rangle$.

The determination of $A'$ requires exhaustive search in general. Thus the computational cost finding $A'$ from the measurement outcomes is large when both $n$ and $d$ are large. However, in certain special cases we can efficiently determine $A'$. An efficient method finding $A'$ is presented in Sect. 3.2.

Remark 3: The error correction method presented in this subsection is not explicitly mentioned in the papers [2], [3], [6]. Still, it can be derived from general facts on quantum error correction presented in [1], [5], [10]. A readable exposition on the error correction procedure is provided by Preskill [16].

3. Construction of Quantum Codes for $p$-State Systems from Classical Codes

3.1 Codes for $p$-State Systems

In this subsection we describe how to construct quantum codes for $p$-state systems from additive codes over $\mathbb{F}_p^2$. Let $\omega$ be a primitive element in $\mathbb{F}_p^2$.

Lemma 4: $\{\omega, \omega^p\}$ is a basis of $\mathbb{F}_p^2$ over $\mathbb{F}_p$.

Proof: When $p = 2$ the assertion is obvious. We assume that $p \geq 3$. Suppose that $\omega^p = \omega$ for some $a \in \mathbb{F}_p$. Then $\omega = \omega^{a^p} = (\omega a)^p = a^2 \omega$, and $a$ is either 1 or $-1$. If $a = 1$, then $\omega \in \mathbb{F}_p$, and $\omega$ is not a primitive element. If $a = -1$, then $\omega^{2p} = \omega^2$. This is a contradiction, because $\omega$ is a primitive element and $2p \neq 2 \pmod{p^2 - 1}$.

For $(a|b) \in \mathbb{F}_p^{2n}$ we define $\phi(a|b) = \omega a + \omega^p b$. Then the weight (3) of $(a|b)$ is equal to the Hamming weight of $\phi(a|b)$. For $c = (c_1, \ldots, c_n), d \in \mathbb{F}_p^{2n}$, we define the inner product of $c$ and $d$ by

$$\langle c, d \rangle - \langle c^p, d \rangle = \langle c, d^p \rangle - \langle c, d^p \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{F}_p^{2n}$ and $c^p = (c_1^p, \ldots, c_n^p)$. For $(a|b), (a'|b') \in \mathbb{F}_p^{2n}$ the inner product (5) of $\phi(a|b)$ and $\phi(a'|b')$ is

$$\langle \phi(a|b), \phi(a'|b') \rangle - \langle \phi(a|b), \phi(a'|b') \rangle = \langle \omega a + \omega^p b, \omega a' + \omega^p b' \rangle$$

$$= \langle \omega (a + a'), \omega^p (b + b') \rangle$$

Since $\omega$ is a primitive element, $\omega^2 \neq \omega^{2p}$. Thus the inner product (1) of $(a|b)$ and $(a'|b')$ is zero iff the inner product (5) of $\phi(a|b)$ and $\phi(a'|b')$ is zero. Thus we have

Theorem 5: Let $C$ be an additive subgroup of $\mathbb{F}_p^{2n}$ containing $p^{n-k}$ elements, $C'$ its orthogonal space with respect to the inner product (5). Suppose that $C' \supset C$ and the minimum Hamming weight of $C' \setminus C$ is $d$. By identifying $\phi^{-1}(C)$ with an Abelian subgroup of $E$ via $X(\cdot)Z(\cdot)$, any eigenspace of $\phi^{-1}(C)$ is an $[[n, k, d]]_p$ quantum code.

# Theorem 5

The map (5) is $\mathbb{F}_p$-bilinear but does not take values in $\mathbb{F}_p$. It is neither $\mathbb{F}_p$-bilinear nor $\mathbb{F}_p^2$-sesquisymmetric. Thus calling the map (5) “inner product” is a little abusive. But the map (5) can be converted to an $\mathbb{F}_p$-bilinear form by dividing it by $\omega^2 - \omega^{2p}$. For this reason we call the map (5) “inner product.”
We next clarify the self-orthogonality of a linear code over $\mathbb{F}_{p^2}$ with respect to $(5)$.

**Lemma 6:** Let $C$ be a linear code over $\mathbb{F}_{p^2}$, and $C'$ the orthogonal space of $C$ with respect to $(5)$. We define $C' = \{ x^p | x \in C \}$ and $(C^p)\perp$ the orthogonal space of $C^p$ with respect to the standard inner product. Then we have $C' = (C^p)\perp$.

**Proof:** It is clear that $C' \supseteq (C^p)\perp$. Suppose that $x \in C'$. Then for all $y \in C$, $\langle x, y^p \rangle = \langle x, y \rangle^p = 0$. Thus $\langle x, y^p \rangle \in \mathbb{F}_p$. Since $\langle x, \omega^p y^p \rangle - \langle x, \omega y^p \rangle^p = 0$, $\omega^p(x, y^p) \in \mathbb{F}_p$. Since $\omega^p \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$, we conclude that $\langle x, y^p \rangle = 0$.

**Theorem 7:** Let $C$ be an $[n, (n - k)/2]$ linear code over $\mathbb{F}_{p^2}$ such that $C \subseteq (C^p)\perp$. Suppose that the minimum Hamming weight of $C$ is $d$. Then any eigenspace of $\phi^{-1}(C)$ is an $[n, k, d]\_p$ quantum code.

### 3.2 Error Correction for $p$-State Systems

In this subsection we consider how to determine the error from measurements with quantum codes obtained via Theorem 7. We retain notations from Theorem 7. Suppose that $g_1, \ldots, g_r$ is an $\mathbb{F}_{p^2}$-basis of $C$. Then $\mathbb{F}_p$-basis of $\phi^{-1}(C)$ is $(a_1|b_1) = \phi^{-1}(g_1), (a_2|b_2) = \phi^{-1}(\omega g_1), \ldots, (a_{2r}|b_{2r}) = \phi^{-1}(\omega^r g_r)$. Suppose that by the procedure in Section 2.3, the error is converted to $A \in E$ that corresponds to $\phi^{-1}(e)$ for some $e \in \mathbb{F}_{p^2}$. We have $\phi^{-1}(A|e|\_p)$ belongs to. By Eq. (2)

$$X(a_i|Z(b_i))A|\_p = X|\_p A|\_p,$$

where $X$ is the alternating inner product $(1)$ of $(a_i|b_i)$ and $\phi^{-1}(e)$, which is denoted by $s_i \in \mathbb{F}_p$. Then we have

$$\langle g_i, e^p \rangle - \langle g_i^p, e \rangle = (\omega^2 - \omega^2p)s_{2i-1},$$

$$\langle \omega g_i, e^p \rangle - \langle \omega g_i^p, e \rangle = (\omega^2 - \omega^2p)s_{2i},$$

It follows that $\langle g_i^p, e \rangle = (\omega^2 - \omega^2p)(s_{2i-1} - s_{2i})/(\omega^p - \omega)$. $\langle g_i^p, e \rangle = (\omega^2 - \omega^2p)/(\omega_{2i-1} - s_{2i})/(\omega^p - \omega)$. $\langle g_i^p, e \rangle$ can be used as rows of the check matrix of $(C^p)\perp$. If we have a classical decoding algorithm for $(C^p)\perp$ finding the error $e$ from a classical syndrome $(g_i^p, e), \ldots, (g_r^p, e)$, then we can find the quantum error $A \in E$.

**Remark 8:** In this section we assumed that $\omega$ is a primitive element in $\mathbb{F}_{p^2}$. It is enough to assume that $\omega$ belongs to $\mathbb{F}_{p^2}$ and $\omega, \omega^p$ are linearly independent over $\mathbb{F}_p$.

### 4. Construction of Quantum Codes for $p^n$-State Systems from Classical Codes

#### 4.1 Codes for $p^n$-State Systems

In this subsection we show a construction of quantum codes for $p^n$-state systems from classical linear codes over $\mathbb{F}_{p^m}$. Our construction is based on the construction [4] by Chen which constructs quantum codes for $2$-state systems from linear codes over $\mathbb{F}_{2^m}$. We modify his construction so that we can estimate the minimum weight $(4)$ from the original code over $\mathbb{F}_{p^m}$.

We fix a normal basis $\{ \theta, \theta^p, \ldots, \theta^{p^m-1} \}$ of $\mathbb{F}_{p^m}$ over $\mathbb{F}_p$. There always exists a normal basis of $\mathbb{F}_{p^m}$ over $\mathbb{F}_p$ [11, Sect. VI,13]. For $a = (a_1, \ldots, a_m, b_1, \ldots, b_m), a' = (a'_1, \ldots, a'_m, b'_1, \ldots, b'_m) \in \mathbb{F}_{p^m}$, we define $\phi(a) = a_1\theta + a_2\theta^p + \cdots + a_m\theta^{p^m-1} + b_1\theta^p + \cdots + b_m\theta^{p^m-1}$, and $T(a, a') = c_{m+1} - 1 \in \mathbb{F}_p$, where $\phi(a)\phi(a')^m = c_1\theta + \cdots + c_{2m}\theta^{p^m-1}$ and $c_i \in \mathbb{F}_p$. Then $T$ is a bilinear form.

**Lemma 9:** $T$ is alternating and nondegenerate.

**Proof:** First we show that $T$ is alternating, that is, $T(a, a) = 0$ for all $a \in \mathbb{F}_{p^m}$. Let $a = \phi(a) = a_1\theta + \cdots + a_m\theta^{p^m-1}$ for $c_i \in \mathbb{F}_p$. Then $(a_1\theta^p)^m = -a_1\theta + \cdots + a_m\theta^{p^m-1} + b_1\theta^p + \cdots + b_m\theta^{p^m-1}$. Since $(a_1\theta^p)^m = c_1\theta + \cdots + c_{2m}\theta^{p^m-1}$ and $c_i \in \mathbb{F}_p$ for $i = 1, \ldots, m$. Hence $T(a, a) = 0$.

We assume that $x \neq 0$, which implies that $a \neq 0$. Since $x(\theta/x^p) = \theta^{p^m}, T(a, \phi^{-1}(\theta/x^p)) = 1$, which shows the nondegeneracy.

**Lemma 10:** By abuse of notation, we denote by $T$ the representation matrix of the bilinear form $T$ with respect to the standard basis of $\mathbb{F}_{p^m}$, that is, for $a, b \in \mathbb{F}_{p^m}$, we have $T(a, b) = a^Tb'$. Let $I_m$ be the $m \times m$ unit matrix and $S = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$. There exists a nonsingular $2m \times 2m$ matrix $D$ such that $DTD' = S$.

**Proof:** See [11, Chapter XV] and use the previous lemma.

For $c = (c_1, \ldots, c_m) \in \mathbb{F}_{p^m}$, let $(a_1, \ldots, a_{i_m}, b_1, \ldots, b_{i_m}) = \phi^{-1}(c_1)D^{-1} \in \mathbb{F}_{p^2}$. We define $\Phi(c) = (a_{i_1}, a_{i_2}, \ldots, a_{i_m}, a_{2i_1}, \ldots, a_{2i_m}, b_{i_1}, \ldots, b_{i_m})$. Then it is clear that the Hamming weight of $c$ is equal to the weight $(4)$ of $\Phi(c)$, since $D$ is a nonsingular matrix. For $a, b \in \mathbb{F}_{p^m}$ we consider the inner product

$$\langle a, b^m \rangle,$$

where $\langle , \rangle$ denotes the standard inner product in $\mathbb{F}_{p^m}$.

**Proposition 11:** Let $C \subseteq \mathbb{F}_{p^m}$ be a linear code over $\mathbb{F}_{p^2}$, and $C'$ the orthogonal space of $C$ with respect to $(6)$. Then the orthogonal space of $\Phi(C)$ with respect to $(1)$ is $\Phi(C')$. 

**Proof:**
Theorem 12: Let \( C \subset \mathbb{F}_{p^m}^n \) be an \( [n, (n-k)/2] \) linear code over \( \mathbb{F}_{p^m} \), \( C^\perp = \{ x^p \mid x \in C \} \), and \( (C^\perp)^\perp \) the orthogonal space of \( C^\perp \) with respect to the standard inner product. Suppose that \( C \subseteq (C^\perp)^\perp \), and the minimum Hamming weight of \( (C^\perp)^\perp \) is \( d \). Then the minimum weight \( (4) \) of \( \Phi(C) \) is \( d \), and \( \Phi(C) \) is self-orthogonal with respect to the inner product \( 1 \). Any eigenspace of \( \Phi(C) \) is an \([n, k, d]_p \) quantum code.

4.2 Error Correction for \( p^m \)-State Systems

In this subsection we consider how to determine the error from measurements with quantum codes obtained via Theorem 12. We retain notations from Theorem 12. Suppose that \( g_1, \ldots, g_m \) is an \( \mathbb{F}_{p^m} \)-basis of \( C \).

Suppose that by a similar procedure to Sect. 2.2, the error is converted to a unitary matrix corresponding to \( \Phi(e) \) for \( e \in \mathbb{F}_{p^m}^n \). We fix a basis \( \{\alpha_1, \ldots, \alpha_{2m}\} \) of \( \mathbb{F}_{p^m}^n \) over \( \mathbb{F}_p \). Then \( \mathbb{F}_p \)-basis of \( \Phi(C) \) is \( \{\Phi(\alpha_j g_i) \mid i = 1, \ldots, r, j = 1, \ldots, 2m\} \). First we shall show how to calculate \( \langle e, g_i^p \rangle \) for each \( i \). For \( j = 1, \ldots, 2m \), let \( (a_j | b_j) = \Phi(\alpha_j g_i) \). As in Sect. 3.2, by the measurement outcomes we can know the inner product \( 1 \) of \( \Phi(e) \) and \( \Phi(\alpha_j g_i) \), denoted by \( s_j \), for \( j = 1, \ldots, 2m \).

For \( x = c_1 \theta + \cdots + c_{2m} \theta^{p^{m-1}} \in \mathbb{F}_{p^m}^n \), we define \( P(x) = c_{m+1} - c_1 \). Then \( P \) is a nonzero \( \mathbb{F}_p \)-linear map. As discussed in the proof of Proposition 11, \( s_j = P(\langle e, \alpha_j g_i^p \rangle) = P(\alpha_j^p \langle e, g_i^p \rangle) \). We define the map \( P_{2m} : \mathbb{F}_{p^2}^m \to \mathbb{F}_{p^m}^n \), \( x \mapsto P(\langle x^p | 1 \rangle, \ldots, \langle x^p | 2m \rangle) \). Then \( P_{2m} \) is an \( \mathbb{F}_p \)-linear map, and \( P_{2m}(\langle e, g_i^p \rangle) = (s_1, \ldots, s_{2m}) \). If \( P_{2m} \) is an isomorphism, then finding \( \langle e, g_i^p \rangle \) from \( (s_1, \ldots, s_{2m}) \) is a trivial task, merely a matrix multiplication. We shall show that \( P_{2m} \) is an isomorphism.

**Lemma 13:** [11, Theorem 6.1, Chapter III] Let \( W \) be a \( 2m \)-dimensional vector space over a field \( K \) with a basis \( \{x_1, \ldots, x_{2m}\} \), and \( \tilde{W} \) the dual of \( W \), that is, the \( K \)-linear space consisting of linear maps from \( W \) to \( K \). Then there exists a basis \( \{f_1, \ldots, f_{2m}\} \) of \( \tilde{W} \) such that \( f_k(x_j) = \delta_{jk} \). \( \{f_1, \ldots, f_{2m}\} \) is called the dual basis.

**Lemma 14:** There exist \( \beta_1, \ldots, \beta_{2m} \in \mathbb{F}_{p^m} \) such that \( P(\alpha_j^p \beta_k) = \delta_{jk} \).

**Proof:** Notice that \( \{\alpha_1^p, \ldots, \alpha_{2m}^p\} \) is an \( \mathbb{F}_p \)-basis of \( \mathbb{F}_{p^m}^p \). The dual space \( \mathbb{F}_{p^m}^p \) can be regarded as an \( \mathbb{F}_p \)-linear space by defining \( x f : u \mapsto f(xu) \) for \( x \in \mathbb{F}_{p^m}^p \) and \( f \in \mathbb{F}_{p^m}^p \). Let \( f_1, \ldots, f_{2m} \) be the dual basis of \( \{\alpha_1^p, \ldots, \alpha_{2m}^p\} \). Since \( \mathbb{F}_{p^m}^p \) is one-dimensional, \( \mathbb{F}_{p^m} \)-linear space and \( 0 \neq P \in \mathbb{F}_{p^m}^p \), \( f_k \) can be written as \( \beta_k P \) for some \( \beta_k \in \mathbb{F}_{p^m} \). It is clear that \( P(\alpha_j^p \beta_k) = \delta_{jk} \).

**Proposition 15:** \( P_{2m} \) is an isomorphism.

**Proof:** It suffices to show that \( P_{2m} \) is surjective. For \( (a_1, \ldots, a_{2m}) \in \mathbb{F}_{p^2}^m \), \( P_{2m}(a_1 \beta_1 + \cdots + a_{2m} \beta_{2m}) \). We have to often puncture a cyclic code length. With their construction we can correct errors up to the BCH bound using the Berlekamp-Massey algorithm.

If we use the Hartmann-Tzeng bound [7] or the restricted shift bound [14] then we get a better estimation of the minimum distance, and we can correct more errors using modified versions of the Feng-Rao decoding algorithm in [14, Theorem 6.8 and Remark 6.12]. The algorithms [14, Theorem 6.8 and Remark 6.12] correct errors up to the Hartmann-Tzeng bound or the restricted shift bound.

We cannot construct good cyclic codes of arbitrary code length. So we have to often puncture a cyclic code as in [3, Theorem 6.6] to get a quantum code with efficient decoding algorithms. In classical error correction, we correct errors of a punctured code by applying an error-and-erasure decoding algorithm for the original code to the received word. But there is no (classical) received word in quantum error correction. So we decode a quantum punctured code as follows: Let \( C' \subset \mathbb{F}_p^d \) be a (classical) linear code, \( h'_1, \ldots, h'_r \), the rows of a check matrix for \( C' \), \( C \) the punctured
code of \(C'\) obtained by discarding the first coordinate, \(h_1, \ldots, h_{r-1}\) the rows of a check matrix for \(C\), and \(0h_i\) the concatenation of 0 and \(h_i\) for \(i = 1, \ldots, r - 1\). We can express \(0h_i\) as

\[
0h_i = \sum_{j=1}^{r} a_{ij} h'_j,
\]

where \(a_{ij} \in \mathbb{F}_q\) [15, Lemma 10.1]. Suppose that an error \(e = (e_2, \ldots, e_n) \in \mathbb{F}_q^{n-1}\) occurs and that we have the syndrome \(s_1 = \langle h_1, e \rangle, \ldots, s_{r-1} = \langle h_{r-1}, e \rangle\). We want to find \(e\) from \(s_1, \ldots, s_{r-1}\) using an error-and-erasure decoding algorithm for \(C'\). We can find \(s'_1, \ldots, s'_r \in \mathbb{F}_q\) such that

\[
s_i = \sum_{j=1}^{r} a_{ij} s'_j
\]

for \(i = 1, \ldots, r - 1\). Then there exists \(e' = (e'_1, \ldots, e'_n) \in \mathbb{F}_q^n\) such that

\[
\langle h'_j, e' \rangle = s'_j \quad \text{for} \quad j = 1, \ldots, r - 1.
\]

because the condition (7) implies that \(\langle 0h_i, e' \rangle = s_i\) for \(i = 1, \ldots, r - 1\). If we apply an error-and-erasure decoding algorithm to the syndrome \(s'_1, \ldots, s'_r\) with the erasure in the first coordinate, then we find \(e'\).

Note that the algorithms [14, Theorem 6.8 and Remark 6.12] are error-only decoding algorithms but we can modify them to error-and-erasure algorithms along the same line as [17, Sect. VI].

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References


\(^\dagger\)The LANL eprints can be obtained from http://xxx.lanl.gov/.


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